Intro (*D*) Problem Sheet

These three exercises are proposed by the lecturers as the ones to concentrate on for the exercise class on Thursday. During the class the students are encouraged to ask about other exercises which were set in the lectures. Those running the class will do their best to explain them, but it should be understood that the explanation may not be coherent, or even bounded.

1.1) *(Introduction to derived categories and stability conditions, Exercise 1.5.3)*
Let $\mathcal{A}$ be an abelian category of global dimension 1, i.e.

$$\mathrm{Ext}^p_{\mathcal{A}}(M,N) = 0 \text{ for all } p > 1 \text{ and for all } M, N \in \mathcal{A}.$$

Prove that every $E \in D^b(\mathcal{A})$ satisfies $E = \bigoplus_{i\in\mathbb{Z}} H^i(E)[-i]$.

1.2) *(Semiorthogonal decomposition of derived categories)*
Compute the mutations $\mathbb{L}_{\mathcal{O}(k)}\mathcal{O}(k+1)$ and $\mathbb{R}_{\mathcal{O}(k+1)}\mathcal{O}(k)$ in $D^b(\mathbb{P}^1)$.
Compute all compositions of at most 3 mutations starting from exceptional collection $\langle \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\rangle$ in $D^b(\mathbb{P}^2)$.

1.3) *(Introduction to DG-categories)*

(a) Prove that representable modules are h-projective. **Extra:** Prove that semifree modules are h-projective.

(b) Prove that $\forall E \in \mathrm{Mod}(\mathcal{A})$ the following are equivalent:

(i) E is h-projective

(ii) For every diagram

$$\begin{array}{ccc} & & F \\ & & \big\downarrow{\scriptstyle \beta} \\ E & \xrightarrow[\alpha]{} & G \end{array}$$

with β a quasi-isomorphism, there exists unique $\gamma \in \mathrm{Hom}_{H^0(\mathrm{Mod}(\mathcal{A})}(E,F)$ such that $\beta \circ \gamma = \alpha$ in $H^0(\mathrm{Mod}(\mathcal{A})$.

(iii) Every quasi-isomorphism $F \to E$ has a left inverse in $H^0(\mathrm{Mod}(\mathcal{A})$.

(c) Define *h-injective* modules and restate (b) for them.